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SOLUTIONS.

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FIND the sum of the series

$$1^2 + 3^2 + 6^2 + 10^2 + 15^2 + \dots + \left[\frac{1}{2} n (n + 1)\right]^2.$$

[*Artemas Martin.*]

SOLUTION.

The fifth differences vanish and the first terms of the difference series are 1, 8, 19, 18, 6, 0 ; hence

$$S = {}_nC_1 + {}_nC_2 \cdot 8 + {}_nC_3 \cdot 19 + {}_nC_4 \cdot 18 + {}_nC_5 \cdot 6,$$

where ${}_nC_i$, etc. are the binomial coefficients. [*Geo. R. Dean.*]

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FOUR equianharmonic points give four triangles which have four circum-circles. Show that the inverses of any point with regard to these four circles are equianharmonic. [*Frank Morley.*]

SOLUTION.

It is proved in Salmon's Conic Sections, § 54, that if the perpendiculars from three points a_1, a_2, a_3 on the sides of a triangle $c_1c_2c_3$ meet at a point a_4 , then also the perpendiculars from c_1, c_2, c_3 on the sides of $a_1a_2a_3$ meet at a point c_4 . And this can be readily proved by elementary geometry, by observing that the condition of either concurrence is, that in the hexagon $a_1, c_3, a_2, c_1, a_3, c_2$ the sum of the squares of three alternate sides is equal to the sum of the squares of the other three sides.

Now

$$\left| \frac{c_2 - c_3}{c_1 - c_3} \right| = \frac{\sin c_2c_1c_3}{\sin c_3c_2c_1} = \frac{\sin a_2a_4a_3}{\sin a_3a_4a_1},$$

$$\left| \frac{c_1 - c_4}{c_2 - c_4} \right| = \frac{\sin c_4c_2c_1}{\sin c_4c_1c_2} = \frac{\sin a_1a_3a_4}{\sin a_4a_3a_2},$$

whence

$$\left| \frac{c_2 - c_3 \cdot c_1 - c_4}{c_1 - c_3 \cdot c_2 - c_4} \right| = \frac{\sin a_2a_4a_3 \cdot \sin a_1a_3a_4}{\sin a_4a_3a_2 \cdot \sin a_3a_4a_1} = \left| \frac{a_2 - a_3 \cdot a_1 - a_4}{a_2 - a_4 \cdot a_1 - a_3} \right|.$$

Hence two corresponding cross-ratios of the two tetrads of points have equal

absolute values. They have also congruent amplitudes ; for

$$\text{am}(c_2 - c_3) \equiv \text{am}(a_1 - a_4) - \pi/2,$$

$$\text{am}(c_1 - c_4) \equiv \text{am}(a_2 - a_3) + \pi/2;$$

and, therefore,

$$\text{am}(c_2 - c_3 \cdot c_1 - c_4) \equiv \text{am}(a_2 - a_3 \cdot a_1 - a_4).$$

Hence corresponding cross-ratios are equal.

In particular, let c_1, c_2, c_3 be the centres of the circles $a_2a_3a_4, a_3a_4a_1, a_4a_1a_2$; then c_4 is the centre of the circle $a_1a_2a_3$; and the theorem is: The inverses of the point ∞ , with regard to the four circles through three of any four points a , have the same cross-ratios as the four points. This is a covariantive statement, in which we can substitute any other point for the point ∞ .

[*Frank Morley.*]

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SHOW that the areas of the curves

$$y = px^2 + qx + r, \text{ and } y = mx^3 + px^2 + qx + r,$$

taken between the limits $x + h$ and $x - h$, are given by the formulæ

$$\Omega = 2yh + \frac{2}{3}ph^3, \text{ and } \Omega = 2yh + (\frac{2}{3}p + 2mx)h^3,$$

respectively.

[*W. H. Echols.*]

SOLUTION I.

From the well-known formula

$$\Omega = \int y dx$$

we have for the first curve

$$\begin{aligned} \Omega &= \int_{x-h}^{x+h} (px^2 + qx + r) dx = \frac{1}{3} px^3 + \frac{1}{2} qx^2 + rx \Big|_{x-h}^{x+h} \\ &= 2h(px^2 + qx + r) + \frac{2}{3} ph^3 = 2yh + \frac{2}{3} ph^3; \end{aligned}$$

and for the second curve

$$\begin{aligned} \Omega &= \int_{x-h}^{x+h} (mx^3 + px^2 + qx + r) dx = \frac{1}{4} mx^4 + \frac{1}{3} px^3 + \frac{1}{2} qx^2 + rx \Big|_{x-h}^{x+h} \\ &= 2h(mx^3 + px^2 + qx + r) + (\frac{2}{3}p + 2mx)h^3 = 2yh + (\frac{2}{3}p + 2mx)h^3. \end{aligned}$$

[*Artemas Martin.*]

SOLUTION II.

By the familiar formula called Simpson's rule the mean value of

$$y = mx^3 + px^2 + qx + r$$

is the sixth part of the sum of the extreme values plus four times the middle value. But

$$(x+h)^3 + (x-h)^3 + 4x^3 = 6x^3 + 6h^2x,$$

$$(x+h)^2 + (x-h)^2 + 4x^2 = 6x^2 + 2h^2,$$

$$(x+h) + (x-h) + 4x = 6x;$$

$$y_m = mx^3 + px^2 + qx + r + mh^2x + \frac{1}{3}ph^2$$

$$= y + h^2(mx + \frac{1}{3}p),$$

and

$$A = 2hy + h^3(2mx + \frac{2}{3}p).$$

Putting $m = 0$, we have the other result.

[*W. M. Thornton.*]

SOLUTION III.

As applications of

$$\frac{1}{2} \int_{x-h}^{x+h} f(x) \cdot dx = hf(x) + \frac{h^3}{3!} f''(x) + \frac{h^5}{5!} f^{(4)}(x) + \dots,$$

we have for the first one,

$$\frac{1}{2} \Omega = hy + \frac{h^3}{3!} 2p;$$

for the second

$$\frac{1}{2} \Omega = hy + \frac{h^3}{3!} (3 \cdot 2mx + 2p). \quad [\textit{W. H. Echols.}]$$

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A WOODEN hemisphere floats in water, vertex down, with $1/n$ of its axis immersed. Find the specific gravity of the hemisphere.

[*Artemas Martin.*]

SOLUTION.

Let s be the specific gravity of the hemisphere and r its radius. The weight of the hemisphere equals the weight of the water displaced, or,

$$\frac{2}{3} \pi r^3 s = \pi \left(\frac{r}{n} \right)^2 \left[r - \frac{1}{3} \cdot \frac{r}{n} \right],$$

which gives

$$s = \frac{3n-1}{2n^3}. \quad [\textit{W. O. Whitescarver.}]$$

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A SOLID sphere and a solid cylinder of equal radii roll from rest down the same inclined plane; compare the times of their descent.

[*Artemas Martin.*]

SOLUTION.

By the principle of vis viva,

$$mv^2 + I\omega^2 = 2mgs \sin i.$$

The motion being pure rolling,

$$v = r\omega.$$

For the sphere,

$$I = \frac{2}{5}mr^2.$$

For the cylinder,

$$I = \frac{1}{2}mr^2.$$

Substituting these values and reducing, we have, for the sphere,

$$v = \frac{ds}{dt} = \sqrt{\frac{5}{14}gs \sin i};$$

and for the cylinder,

$$v = \frac{ds}{dt} = \sqrt{\frac{1}{3}gs \sin i}.$$

Integrating, we get

$$\sqrt{s} = t \sqrt{\frac{5}{14}g \sin i}, \quad \sqrt{s} = t' \sqrt{\frac{1}{3}g \sin i};$$

whence

$$\frac{t}{t'} = \sqrt{\frac{14}{5}}.$$

[*S. T. Moreland; M. C. Andrews; C. E. Mendenhall; G. R. Dean.*]

EXERCISE.

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GIVEN $\tan_{\kappa} w = x + iy$, wherein

$$\tan_{\kappa} w = x(e^{w/\kappa} - e^{-w/\kappa})/(e^{w/\kappa} + e^{-w/\kappa}),$$

and

$$w = u + iv, \quad i = \sqrt{-1},$$

$$x = m(\cos \beta + i \sin \beta);$$

determine u and v as real quantities in terms of x , y , m , and β .

[*Irving Stringham.*]